

An 8-dimensional realization of the Clifford algebra in the 5-dimensional Galilean space-time ¹

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Abstract

We give an 8-dimensional realization of the Clifford algebra in the 5-dimensional Galilean space-time by using a dimensional reduction from the (5+1) Minkowski space-time to the (4+1) Minkowski space-time which encompasses the Galilean space-time. A set of solutions of the Dirac-type equation in the 5-dimensional Galilean space-time is obtained, based on the Pauli representation of 8×8 gamma matrices. In order to find an explicit solution, we diagonalize the Klein-Gordon divisor by using the Galilean boost.

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1 Introduction

Nearly twenty years ago, Takahashi investigated the reduction from a $(4+1)$ Galilean covariant manifold to the Newtonian space-time (with 3-dimensional space) as means to build non-relativistic many-body theories by starting with Lorentz-like, manifestly covariant, equations [1]. This ‘Galilean manifold’ is actually a $(4+1)$ Minkowski space-time with light-cone coordinates, which is reduced to the usual Newtonian space-time [2]. Galilean covariant theories for the Dirac-type fields have been developed by using a 4-dimensional realization of the Clifford algebra in a 5-dimensional Galilean space-time [2]. Therein, we have 16 independent components that may be expressed as $\gamma_A = \mathbb{I}, \gamma_\mu, \sigma_{\mu\nu}$, with $\mu, \nu = 1, \dots, 5$ [3]. Unfortunately, none of the pseudo-tensor interactions of rank 0, 1 and 2 can be introduced into 5- (or any odd-) dimensional theories, because they admit no ‘ γ^6 matrix’ which corresponds to the γ^5 of the $(3+1)$ Minkowski space-time. A 4-dimensional realization of the Clifford algebra in the $(4+1)$ Minkowski space-time requires γ^5 as a fourth spatial element of γ_μ s. Motivated by this fact, we discuss in this article an 8-dimensional realization of the Clifford algebra in the $(4+1)$ Galilean space-time. Thus our formulation involves two successive dimensional reductions: from the $(5+1)$ Minkowski space-time to a $(4+1)$ Minkowski space-time, which corresponds to the 5-dimensional Galilean extended space-time mentioned earlier, and then from this extended manifold to the usual Newtonian space-time [4].

Parity refers to a reversal of orientation of the spatial manifold. This corresponds to the reversal of coordinates in even-dimensional Minkowski space-times. In odd-dimensional space-times, in which the number of spatial coordinates is even, the reflection of spatial manifold has determinant equal to one and hence it is continuously connected to the identity, and so can be obtained as a rotation. Therefore, we must define parity as the reversal of sign of an odd number of spatial coordinates in order to reverse the orientation of the spatial volume. This is the reason why we start in the $(5+1)$ Minkowski space-time in order to define parity operation in the $(4+1)$ Galilean space-time.

The development of 8-dimensional gamma matrices for the Dirac equation is motivated by applications to problems like the beta decay in the 4-fermion Lagrangian of the $V - A$ theory. This requires an evaluation of operators like

$$\bar{\psi}_h \gamma_\mu (1 - \gamma^5) \psi_h \psi_l \gamma^\mu (1 - \gamma^5) \psi_l,$$

which are a combination of the hadron and lepton currents. Hence the necessity to have a γ^5 matrix which provides us with a chirality operator. The leptonic part will be Poincaré invariant and the hadronic part will be Galilean invariant. The simplest example is the neutron decay:

$$n \longrightarrow p + e^- + \bar{\nu}.$$

This will provide us with an amplitude that still possesses a symmetry instead of just

using an expansion in terms of p/m , thus destroying any symmetry in the hadronic part.

Another application of γ^5 matrices is in deriving an $N-N$ potential with a pseudo-vector or pseudo-scalar coupling. Although there is no Yukawa coupling in Galilean covariant theories, it is still possible to define a 4-point coupling. This will provide an analogue of the π -meson exchange $N-N$ potential in the Poincaré-invariant theories. In addition, it is obvious that the interaction term has similarities with the Nambu-Jona-Lasinio theory [5]. Such a development may also be followed in order to obtain further results for the strongly interacting hadronic systems. Our purpose is to make progress along these lines with a Galilean covariant theory in $(4+1)$ space-time. However, in order to define the γ^5 -like matrix, it is necessary to further extend the theories to a $(5+1)$ Minkowski manifold. Results of this article are therefore quite important in order to gain an understanding of the associated physical phenomena.

In section 2, we give an 8-dimensional realization of the Clifford algebra in the $(5+1)$ Minkowski space-time. Then, in section 3, we construct wave functions for the Dirac equation in this space-time. By dimensional reduction from the $(5+1)$ Minkowski space-time to the $(4+1)$ Minkowski space-time, we obtain 8×8 gamma matrices obeying the Clifford algebra in the $(4+1)$ Galilean space-time in section 4. The construction of wave functions for the Dirac-type equation in the $(4+1)$ Galilean space-time is performed in section 5. The final section contains concluding remarks.

We establish the commutation and anticommutation relations of 8×8 gamma matrices in appendix A, and their trace formulas in appendix B. Fierz identities are developed in appendix C. Finally, in appendix D, we give explicit forms of wave functions obtained in sections 3 and 5. Throughout this work, we use the natural units, in which $\hbar = 1$ and $c = 1$.

2 An 8-dimensional realization of the Clifford algebra in the $(5+1)$ Minkowski space-time

Before we turn to the 8-dimensional realization, we begin this section by recalling some useful properties of representations of the gamma matrices. The γ -matrices obey the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (1)$$

where we choose the metric tensor to be given by

$$g_{\mu\nu} = \text{diag}(1, 1, 1, 1, 1, -1) = g^{\mu\nu}, \quad (2)$$

such that

$$g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu.$$

If γ^μ and γ'^μ are two irreducible sets of gamma matrices which satisfy the Clifford algebra given in Eq. (1), then there exists a non-singular matrix S such that

$$\gamma'^\mu = -S^{-1}\gamma^\mu S.$$

By taking this matrix to be equal to

$$S = \gamma^0 = -\gamma_0,$$

where $(\gamma^0)^2 = -1$ and $(\gamma^0)^{-1} = -\gamma^0 = \gamma_0$, we can define the Hermitian conjugate of γ^μ as follows:

$$(\gamma^\mu)^\dagger = -(\gamma^0)^{-1} \gamma^\mu \gamma^0 = \gamma^0 \gamma^\mu \gamma^0.$$

This reads explicitly as

$$\begin{aligned} (\gamma^i)^\dagger &= \gamma^i, & (i = 1, 2, 3), \\ (\gamma^a)^\dagger &= \gamma^a, & (a = 4, 5), \\ (\gamma^0)^\dagger &= -\gamma^0 = (\gamma^0)^{-1}. \end{aligned}$$

The transposition of gamma matrices is defined by taking S to be equal to

$$S = C = -\gamma^0 \hat{C}, \quad (3)$$

so that the transpose is obtained as follows:

$$\begin{aligned} (\gamma^\mu)^T &= -C^{-1} \gamma^\mu C = -C^{-1} \gamma^0 (\gamma^0 \gamma^\mu \gamma^0) \gamma^0 C, \\ &= C^{-1} (\gamma^0)^{-1} (\gamma^\mu)^\dagger \gamma^0 C, \\ &= \hat{C}^{-1} (\gamma^\mu)^\dagger \hat{C}. \end{aligned} \quad (4)$$

Note that

$$\gamma^\mu = \hat{C} (\gamma^\mu)^* \hat{C}^{-1},$$

where, with $*$ denoting the complex conjugation,

$$\hat{C}^\dagger = \hat{C}^{-1} = -\hat{C}^*.$$

2.1 An 8-dimensional realization of the Clifford algebra

In order to obtain an explicit form of 8×8 gamma matrices in a 6-dimensional space-time, let us introduce the following nine matrices:

$$\begin{aligned} \rho &= \sigma \otimes I \otimes I, \\ \pi &= I \otimes \sigma \otimes I, \\ \Sigma &= I \otimes I \otimes \sigma, \end{aligned} \quad (5)$$

where σ are the Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the matrices defined in Eq. (5) are

$$\begin{aligned}\rho_1 &= \begin{pmatrix} \mathbf{0}_{4 \times 4} & I & 0 \\ I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & I & \mathbf{0}_{4 \times 4} \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \mathbf{0}_{4 \times 4} & -iI & 0 \\ 0 & -iI & \mathbf{0}_{4 \times 4} \\ iI & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & I & -I \\ \mathbf{0}_{4 \times 4} & -I & 0 \end{pmatrix}, \\ \pi_1 &= \begin{pmatrix} 0 & I & \mathbf{0}_{4 \times 4} \\ I & 0 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & 0 & I \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 0 & -iI & \mathbf{0}_{4 \times 4} \\ iI & 0 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & 0 & -iI \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & -I & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & I & 0 \end{pmatrix}, \\ \Sigma &= \begin{pmatrix} \sigma & 0 & \mathbf{0}_{4 \times 4} \\ 0 & \sigma & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \sigma & 0 \end{pmatrix}.\end{aligned}$$

The following relations hold among these matrices:

$$\begin{aligned}[\rho_k, \Sigma_l] &= [\pi_k, \Sigma_l] = [\rho_k, \pi_l] = 0, \\ \rho_k \rho_l &= \delta_{kl} + i\epsilon_{klm} \rho_m, \\ [\rho_k, \rho_l] &= 2i\epsilon_{klm} \rho_m, \\ \{\rho_k, \rho_l\} &= 2\delta_{kl}, \quad k, l, m = 1, 2, 3,\end{aligned}$$

with similar relations for the π s and Σ s.

To complete our construction, we introduce three mutually orthogonal unit vectors, \mathbf{m} , \mathbf{n} , and \mathbf{l} , which are utilized to express the gamma matrices as follows:

$$\begin{aligned}\mathbf{m} \cdot \rho &= i\gamma^0, \\ (\mathbf{m} \times \mathbf{n}) \cdot \rho &= \gamma^7, \\ (\mathbf{n} \cdot \rho) (\mathbf{m} \cdot \pi) &= \gamma^4, \\ (\mathbf{n} \cdot \rho) (\mathbf{l} \cdot \pi) &= \gamma^5, \\ (\mathbf{n} \cdot \rho) (\mathbf{n} \cdot \pi) \Sigma &= \gamma.\end{aligned}\tag{6}$$

We can prove that the γ^μ s given by these equations satisfy the Clifford algebra (1), and that γ^7 can be cast in the following form:

$$\gamma^7 = \frac{1}{6!} \epsilon_{\mu\nu\lambda\rho\sigma\tau} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma^\sigma \gamma^\tau = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^0,$$

with

$$\epsilon_{012345} = -\epsilon^{012345} = -\epsilon_{123450} = -1.$$

Commutation and anticommutation relations involving the γ -matrices in a $(5+1)$ Minkowski manifold are given in appendix A.1, and the corresponding trace relations are in appendix B.1.

In the next section, we discuss specific definitions of \mathbf{m} , \mathbf{n} and \mathbf{l} which, in turn, lead to a particular representation of the Dirac matrices.

2.2 The Pauli representation of gamma matrices

In this section we construct a representation, in which $i\gamma^0$ is diagonal, that we shall refer to as the ‘Pauli representation’ of γ -matrices. It is obtained by choosing

$$\mathbf{m} = (0, 0, 1), \quad \mathbf{n} = (0, 1, 0), \quad \mathbf{l} = (1, 0, 0).$$

Therefore, we find

$$\begin{aligned} i\gamma^0 &= \rho_3 = \sigma_3 \otimes I \otimes I = \begin{pmatrix} I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & I & \\ \mathbf{0}_{4 \times 4} & -I & 0 \end{pmatrix}, \\ \gamma^7 &= -\rho_1 = -\sigma_1 \otimes I \otimes I = \begin{pmatrix} & -I & 0 \\ \mathbf{0}_{4 \times 4} & 0 & -I \\ -I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & -I & \end{pmatrix}, \\ \gamma^4 &= \rho_2 \pi_3 = \sigma_2 \otimes \sigma_3 \otimes I = \begin{pmatrix} & -iI & 0 \\ \mathbf{0}_{4 \times 4} & 0 & iI \\ iI & 0 & \mathbf{0}_{4 \times 4} \\ 0 & -iI & \end{pmatrix}, \\ \gamma^5 &= \rho_2 \pi_1 = \sigma_2 \otimes \sigma_1 \otimes I = \begin{pmatrix} & 0 & -iI \\ \mathbf{0}_{4 \times 4} & -iI & 0 \\ 0 & iI & \mathbf{0}_{4 \times 4} \\ iI & 0 & \end{pmatrix}, \\ \gamma &= \rho_2 \pi_2 \Sigma = \sigma_2 \otimes \sigma_2 \otimes \sigma = \begin{pmatrix} & 0 & -\sigma \\ \mathbf{0}_{4 \times 4} & \sigma & 0 \\ 0 & \sigma & \mathbf{0}_{4 \times 4} \\ -\sigma & 0 & \end{pmatrix}. \end{aligned} \tag{7}$$

Note that this representation is equivalent to the one described in Refs. [6, 7]. We can prove it by choosing the following representation:

$$\mathbf{m} = (1, 0, 0), \quad \mathbf{n} = (0, 0, 1), \quad \mathbf{l} = (0, 1, 0),$$

which leads to

$$\begin{aligned} i\gamma^0 &= \rho_1 = \Sigma_1^{(3)}, \\ \gamma^7 &= -\rho_2 = -\Sigma_2^{(3)}, \\ \gamma^4 &= \rho_3 \pi_1 = \Sigma_3^{(3)}, \\ \gamma^5 &= \rho_3 \pi_2 = \Sigma_4^{(3)}, \\ \gamma^k &= \rho_3 \pi_3 \Sigma_k = \Sigma_{4+k}^{(3)}, \quad (k = 1, 2, 3), \end{aligned}$$

where $\Sigma_a^{(3)}$ ($a = 1, \dots, 7$) is in the notation defined in Eq. (4.1) of Ref. [6].

2.3 Number of independent gamma matrices

Let n be the dimension of space-time, so that the number of Γ s is 2^n . Since we have

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k = \sum_{k=0}^n {}_nC_k x^k,$$

and a completely antisymmetric tensor of rank k has ${}_nC_k$ independent elements, then the number of independent Γ s is

$${}_nC_0 + {}_nC_1 + \cdots + {}_nC_n = \sum_{k=0}^n {}_nC_k (1)^k = (1+1)^n = 2^n.$$

Thus there exists 2^n linearly independent matrices:

$$\Gamma_{\mu_1 \cdots \mu_k}^{(k)} = d_{\mu_1 \cdots \mu_k, \nu_1 \cdots \nu_k}^{(k)} \gamma^{\nu_1} \cdots \gamma^{\nu_k}, \quad (k = 0, 1, \dots, n),$$

where $d^{(k)}$ are operators, described in Ref. [8], which project out the totally antisymmetric part of a rank- k tensor.

In the case of 6-dimensional Minkowski space-time, we have $2^6 = 64$ independent gamma matrices, which we write as

$$\begin{aligned} \Gamma^{(0)} &= \mathbb{I}, & (\mathbb{I} \text{ is the } 8 \times 8 \text{ unit matrix}), \\ \Gamma_{\mu}^{(1)} &= \gamma_{\mu}, \\ \Gamma_{\mu\nu}^{(2)} &= \sigma_{\mu\nu} = \frac{1}{2i} (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu}), \\ \Gamma_{\mu\nu\lambda}^{(3)} &= \sigma_{\mu\nu\lambda} = \frac{1}{3} (\gamma_{\mu} \sigma_{\nu\lambda} + \gamma_{\nu} \sigma_{\lambda\mu} + \gamma_{\lambda} \sigma_{\mu\nu}) = -\frac{1}{6} \epsilon_{\mu\nu\lambda\rho\sigma\tau} \sigma^{\rho\sigma\tau} \gamma^7, \\ \Gamma_{\mu\nu\rho\sigma}^{(4)} &= -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma\xi\eta} \sigma^{\xi\eta} \gamma^7, \\ \Gamma_{\mu\nu\rho\sigma\lambda}^{(5)} &= -\epsilon_{\mu\nu\rho\sigma\lambda\xi} \gamma^{\xi} \gamma^7, \\ \Gamma_{\mu\nu\rho\sigma\lambda\tau}^{(6)} &= -\epsilon_{\mu\nu\rho\sigma\lambda\tau} \gamma^7. \end{aligned}$$

To show properties under the Lorentz transformations, we choose the following 64 linearly independent matrices:

$$\gamma_A = \mathbb{I}, \gamma^7, \gamma_{\mu}, i\gamma^7 \gamma_{\mu}, \sigma_{\mu\nu}, \gamma^7 \sigma_{\mu\nu}, \sigma_{\mu\nu\lambda},$$

satisfying

$$\gamma^A \gamma_A = \mathbb{I}, \quad (\text{no summation over } A),$$

$$\text{Tr}(\gamma_A) = 0, \quad \text{if } \gamma_A \neq \mathbb{I},$$

as well as

$$\text{Tr}(\gamma^A \gamma_B) = 8\delta^A_B.$$

By using the charge conjugation matrix C of Eq. (3), we can separate the γ_A s into symmetric and antisymmetric elements as

$$(\gamma_A C)_{\alpha\beta} = \epsilon_A (\gamma_A C)_{\beta\alpha}, \tag{8}$$

or, equivalently,

$$(\gamma_A)_\alpha^\beta = \epsilon_A (C^{-1} \gamma_A C)^\beta_\alpha,$$

where

$$\epsilon_A = \begin{cases} +1 & \text{for } C, \gamma^7 \sigma_{\mu\nu} C, \sigma_{\lambda\mu\nu} C, \\ -1 & \text{for } \gamma^7 C, \gamma_\mu C, i\gamma^7 \gamma_\mu C, \sigma_{\mu\nu} C, \end{cases} \quad (9)$$

We have used the relation $C^\dagger = C^{-1} = C^*$.

Note that the lowercase indices from the beginning of the Greek alphabet, α, β, γ , etc. denote spinor indices, and the lowercase indices from the middle of the alphabet, ξ, κ, λ , etc. are tensor indices.

2.4 Parity

The parity matrix, denoted Π , is defined by imposing that the equation of motion be invariant under the discrete transformation of space reflection:

$$x^\mu \rightarrow x'^\mu = (-\mathbf{x}, x^4, x^5, x^0).$$

Consider the Dirac field, then the requirement reads

$$\eta^{-1} \Pi^\dagger \eta \gamma^\mu \Pi = \begin{cases} -\gamma^\mu, & \text{for } \mu = 1, 2, 3, \\ \gamma^\mu, & \text{for } \mu = 4, 5, 0, \end{cases} \quad (10)$$

where the matrix η is defined as

$$\eta = i\gamma^0. \quad (11)$$

Hence the Dirac equation is invariant under the space reflection.

The parity matrix may be expressed by

$$\Pi = \gamma^4 \gamma^5 \gamma^0.$$

3 Construction of wave functions for the Dirac equation in the $(5 + 1)$ Minkowski space-time

In this section, we obtain the wave functions for the Dirac equation of motion in the extended $(5 + 1)$ Minkowski manifold. We adopt the methods of constructing wave functions developed by Takahashi [9], in which the Klein-Gordon divisor is diagonalized by using the Lorentz boost.

The Dirac equation for massive particles with mass m is expressed in the form:

$$\Lambda(\partial)\psi(x) = 0, \quad (12)$$

where the operator $\Lambda(\partial)$ is given by

$$\Lambda(\partial) = -(\gamma \cdot \partial + m).$$

Here, the scalar product is denoted by $A \cdot B$ and defined by

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = A^i B^i + A^a B^a - A^0 B^0,$$

where the lowercase indices from the beginning of the Latin alphabet, a, b, c , etc. take the values 4 and 5, and the lowercase indices from the middle of the Latin alphabet, i, j, k , etc. run from 1 to 3.

The adjoint equation to Eq. (12) is obtained by taking its Hermitian conjugate:

$$\bar{\psi}(x) \Lambda(-\overleftarrow{\partial}) = 0,$$

($\overleftarrow{\partial}$ denotes the left-derivative) with

$$\bar{\psi}(x) = \psi^\dagger(x) \eta.$$

We assume the existence of a non-singular matrix η which satisfies the relation:

$$[\eta \Lambda(\partial)]^\dagger = \eta \Lambda(-\partial). \quad (13)$$

This condition is equivalent to requiring the hermiticity of the Lagrangian in the form

$$\mathcal{L}(x) = \bar{\psi}(x) \Lambda(\partial) \psi(x).$$

Thus we choose η as in Eq. (11).

The operator $d(\partial)$, reciprocal to the operator $\Lambda(\partial)$ of Eq. (12), is defined by

$$\Lambda(\partial) d(\partial) = d(\partial) \Lambda(\partial) = (\partial^2 - m^2) \mathbb{I}.$$

This reciprocal operator is called the ‘Klein-Gordon divisor’. It is given by

$$d(\partial) = -(\gamma \cdot \partial - m).$$

The Dirac field $\psi(x)$ and its charge-conjugate field $\psi_C(x)$ can be expanded in terms of c -number wave functions with positive and negative frequencies, represented by $u_p^{(r)}(x)$ and $v_p^{(r)}(x)$, respectively, and two kinds of creation and annihilation operators:

$$\begin{aligned} \psi(x) &= \sum_r \int d\mathbf{p} \, d^2 p^a [u_p^{(r)}(x) a^{(r)}(\mathbf{p}, p^a) + v_p^{(r)}(x) b^{(r)\dagger}(\mathbf{p}, p^a)], \\ \psi_C(x) &:= \hat{C} \psi^*(x), \\ &= \sum_r \int d\mathbf{p} \, d^2 p^a [u_p^{(r)}(x) b^{(r)}(\mathbf{p}, p^a) + v_p^{(r)}(x) a^{(r)\dagger}(\mathbf{p}, p^a)], \end{aligned}$$

where

$$\begin{aligned} \{a^{(r)}(\mathbf{p}, p^a), a^{(r')\dagger}(\mathbf{p}', p'^a)\} &= \delta_{rr'} \delta(\mathbf{p} - \mathbf{p}') \delta^{(2)}(p^a - p'^a), \\ \{b^{(r)}(\mathbf{p}, p^a), b^{(r')\dagger}(\mathbf{p}', p'^a)\} &= \delta_{rr'} \delta(\mathbf{p} - \mathbf{p}') \delta^{(2)}(p^a - p'^a), \end{aligned}$$

and all other commutators of similar type vanish. We use the notation

$$d^2 p^a = dp^4 dp^5 \quad \text{and} \quad \delta^{(2)}(p^a - p'^a) = \delta(p^4 - p'^4) \delta(p^5 - p'^5).$$

The function $v_p^{(r)}(x)$ is defined by

$$v_p^{(r)}(x) = \hat{C} u_p^{(r)*}(x).$$

The charge conjugation matrix \hat{C} , defined by Eq. (4), satisfies

$$[\eta\Lambda(\partial)]^T = [\eta\Lambda(-\partial)]^* = \hat{C}^{-1} \eta\Lambda(-\partial) \hat{C}.$$

It is convenient to take the functions $u_p^{(r)}(x)$ to be eigenvectors of the operator $-\mathbf{i}\partial_\mu$:

$$-\mathbf{i}\partial_\mu u_p^{(r)}(x) = p_\mu u_p^{(r)}(x)$$

By substituting the Fourier transform of $u_p^{(r)}(x)$ into this equation, we find

$$u_p^{(r)}(x) = f_p(x) u^{(r)}(\mathbf{p}, p^a),$$

$$v_p^{(r)}(x) = f_p^*(x) v^{(r)}(\mathbf{p}, p^a),$$

where

$$f_p(x) = (2\pi)^{-5/2} e^{\mathbf{i}p \cdot x},$$

and

$$p^0 = \sqrt{\mathbf{p} \cdot \mathbf{p} + (p^4)^2 + (p^5)^2 + m^2}.$$

By following the prescription developed in chapter 5 of Ref. [9], we obtain the orthonormality condition and the closure properties in the momentum representation:

$$\bar{u}^{(r')}(\mathbf{p}, p^a) \mathbf{i}\gamma^0 u^{(r)}(\mathbf{p}, p^a) = \delta_{rr'},$$

$$\bar{v}^{(r')}(\mathbf{p}, p^a) \mathbf{i}\gamma^0 v^{(r)}(\mathbf{p}, p^a) = \delta_{rr'},$$

$$\sum_r u_\alpha^{(r)}(\mathbf{p}, p^a) \bar{u}^{(r)\beta}(\mathbf{p}, p^a) = \frac{1}{2p^0} d_\alpha^\beta(\mathbf{i}p),$$

$$\sum_r v_\alpha^{(r)}(\mathbf{p}, p^a) \bar{v}^{(r)\beta}(\mathbf{p}, p^a) = -\frac{1}{2p^0} d_\alpha^\beta(-\mathbf{i}p).$$

Consider a Lorentz transformation matrix $L(\mathbf{p}, p^a)$ given by

$$L(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2m}} \mathbb{I} - \frac{1}{\sqrt{2m(p^0 + m)}} \gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + p^b \gamma^b). \quad (14)$$

Then we have

$$L^{-1}(\mathbf{p}, p^a) \gamma^\mu L(\mathbf{p}, p^a) = \Lambda^\mu_\nu(\mathbf{p}, p^a) \gamma^\nu, \quad (15)$$

where

$$\begin{aligned}\Lambda^0_{\nu}(\mathbf{p}, p^a) &= \left(\frac{p^k}{m}, \frac{p^a}{m}, \frac{p^0}{m} \right), \\ \Lambda^i_{\nu}(\mathbf{p}, p^a) &= \left(g^{ik} + \frac{p^i p^k}{m(p^0 + m)}, \frac{p^i p^a}{m(p^0 + m)}, \frac{p^i}{m} \right), \\ \Lambda^b_{\nu}(\mathbf{p}, p^a) &= \left(\frac{p^b p^k}{m(p^0 + m)}, g^{ba} + \frac{p^b p^a}{m(p^0 + m)}, \frac{p^b}{m} \right).\end{aligned}$$

The transformation coefficients Λ^μ_{ν} satisfy the relation

$$g_{\mu\nu} \Lambda^\mu_{\rho}(\mathbf{p}, p^a) \Lambda^\nu_{\sigma}(\mathbf{p}, p^a) = g_{\rho\sigma},$$

as is expected, and hence they induce the homogenous Lorentz transformation. It follows from Eq. (15) that

$$L^{-1}(\mathbf{p}, p^a) d(ip) L(\mathbf{p}, p^a) = m(\mathbb{I} + i\gamma^0). \quad (16)$$

The factor $(\mathbb{I} + i\gamma^0)$ plays a crucial role when constructing wave functions, because we find the following key relations from this factor:

$$\begin{aligned}L(\mathbf{p}, p^a)(\mathbb{I} + i\gamma^0) &= \frac{1}{\sqrt{2m(p^0 + m)}} d(ip)(\mathbb{I} + i\gamma^0), \\ (\mathbb{I} + i\gamma^0)L^\dagger(\mathbf{p}, p^a)\eta &= (\mathbb{I} + i\gamma^0)L^{-1}(\mathbf{p}, p^a),\end{aligned}$$

where we have used the relation

$$\gamma^0 L^\dagger(\mathbf{p}, p^a) \gamma^0 = -L^{-1}(\mathbf{p}, p^a).$$

Note that Eq. (4) leads to the useful relation:

$$\hat{C} L^*(\mathbf{p}, p^a) \hat{C}^{-1} = L(\mathbf{p}, p^a).$$

If we choose the Pauli representation for the gamma matrices, the relation (16) states that the Klein-Gordon divisor is diagonalized by the Lorentz boost (14).

The helicity operator h is defined in terms of the rank-3 Pauli-Lubanski tensor:

$$h = -\frac{1}{2} \frac{1}{|\mathbf{p}|} w_{045} = \frac{1}{2} \Sigma_k \frac{p^k}{|\mathbf{p}|}, \quad (17)$$

where Σ_k is defined in Eq. (5), and the complete Pauli-Lubanski tensor is given by

$$w_{\lambda\mu\nu} = \frac{1}{2} \epsilon_{\lambda\mu\nu\rho\alpha\tau} p^\rho \sigma^{\alpha\tau}.$$

By using the representation of gamma matrices given in Eq. (6), we find

$$[L(\mathbf{p}, p^a), h] = 0.$$

To diagonalize the helicity operator (17), we introduce the following unitary matrix:

$$S(\mathbf{p}) = \frac{1}{\sqrt{2(1+n^3)}} [(1+n^3)\mathbb{I} + in^2\Sigma_1 - in^1\Sigma_2], \quad (18)$$

where

$$n^k = \frac{p^k}{|\mathbf{p}|}.$$

By using the matrix (18), we can prove the relation

$$S^{-1}(\mathbf{p})hS(\mathbf{p}) = \frac{1}{2}\Sigma_3,$$

where Σ_3 is defined in Eq. (6), hence the helicity operator h is diagonalized by $S(\mathbf{p})$. The definition in Eq. (18) implies that

$$\hat{C}S^*(\mathbf{p})\hat{C}^{-1} = S(\mathbf{p}).$$

By noticing that

$$\hat{C}h^*\hat{C}^{-1} = -h,$$

we find

$$hu^{(r)}(\mathbf{p}, p^a) = \frac{1}{2}\epsilon^{(r)}u^{(r)}(\mathbf{p}, p^a),$$

and

$$hv^{(r)}(\mathbf{p}, p^a) = h\hat{C}u^{(r)*}(\mathbf{p}, p^a) = -\frac{1}{2}\epsilon^{(r)}v^{(r)}(\mathbf{p}, p^a),$$

in the Pauli representation, where r runs from 1 to 4, and $\epsilon^{(r)}$ is given by

$$\epsilon^{(r)} = \begin{cases} 1 & \text{for } r = 1, 3, \\ -1 & \text{for } r = 2, 4. \end{cases}$$

Wave functions are constructed in the Pauli representation for gamma matrices as follows:

$$\begin{aligned} h_\alpha^\beta u_\beta^{(r)}(\mathbf{p}, p^a) &= \frac{1}{2}\epsilon^{(r)}u_\alpha^{(r)}(\mathbf{p}, p^a), \\ &= \frac{1}{\sqrt{4mp^0}}h_\alpha^\beta [d(ip)L(\mathbf{p}, p^a)S(\mathbf{p})]_\beta^r, \\ &= \sqrt{\frac{m}{p^0}}h_\alpha^\beta [L(\mathbf{p}, p^a)\frac{1}{2}(\mathbb{I} + i\gamma^0)S(\mathbf{p})]_\beta^r, \\ &= \frac{1}{\sqrt{2p^0(p^0+m)}} [(-i\gamma \cdot p + m)\frac{1}{2}(\mathbb{I} + i\gamma^0)hS(\mathbf{p})]_\alpha^r, \\ &= \frac{1}{2}\frac{1}{\sqrt{2p^0(p^0+m)}} [(-i\gamma \cdot p + m)\frac{1}{2}(\mathbb{I} + i\gamma^0)S(\mathbf{p})\Sigma_3]_\alpha^r, \\ \bar{u}^{(r)\beta}(\mathbf{p}, p^a)h_\beta^\alpha &= \frac{1}{2}\epsilon^{(r)}\bar{u}^{(r)\alpha}(\mathbf{p}, p^a), \\ &= \sqrt{\frac{m}{p^0}}[S^{-1}(\mathbf{p})\frac{1}{2}(\mathbb{I} + i\gamma^0)L^{-1}(\mathbf{p}, p^a)]_r^\beta h_\beta^\alpha, \\ &= \frac{1}{2}\frac{1}{\sqrt{2p^0(p^0+m)}} [\Sigma_3S^{-1}(\mathbf{p})\frac{1}{2}(\mathbb{I} + i\gamma^0)(-i\gamma \cdot p + m)]_r^\alpha, \end{aligned} \quad (19)$$

$$\begin{aligned}
h_\alpha^\beta v_\beta^{(r)}(\mathbf{p}, p^a) &= -\frac{1}{2}\epsilon^{(r)} v_\alpha^{(r)}(\mathbf{p}, p^a), \\
&= \sqrt{\frac{m}{p^0}} h_\alpha^\beta [L(\mathbf{p}, p^a) \frac{1}{2}(\mathbb{I} - i\gamma^0) S(\mathbf{p}) \Sigma_3 \hat{C}]_{\beta r}, \\
&= \frac{1}{2} \frac{1}{\sqrt{2p^0(p^0+m)}} [(i\gamma \cdot p + m) \frac{1}{2}(\mathbb{I} - i\gamma^0) S(\mathbf{p}) \Sigma_3 \hat{C}]_{\alpha r}, \\
\bar{v}^{(r)\beta}(\mathbf{p}, p^a) h_\beta^\alpha &= -\frac{1}{2}\epsilon^{(r)} \bar{v}^{(r)\alpha}(\mathbf{p}, p^a), \\
&= -\sqrt{\frac{m}{p^0}} [\hat{C}^{-1} S^{-1}(\mathbf{p}) \frac{1}{2}(\mathbb{I} - i\gamma^0) L^{-1}(\mathbf{p}, p^a)]^{r\beta} h_\beta^\alpha, \\
&= -\frac{1}{2} \frac{1}{\sqrt{2p^0(p^0+m)}} [\hat{C}^{-1} \Sigma_3 S^{-1}(\mathbf{p}) \frac{1}{2}(\mathbb{I} - i\gamma^0) (i\gamma \cdot p + m)]^{r\alpha},
\end{aligned} \tag{20}$$

where the charge conjugation matrix, obtained from Eq. (4), is given by

$$\hat{C} = \gamma^0 \gamma^4 \gamma^5 \gamma^2 = \begin{pmatrix} \mathbf{0}_{4 \times 4} & -i\sigma_2 & 0 \\ -i\sigma_2 & 0 & -i\sigma_2 \\ 0 & -i\sigma_2 & \mathbf{0}_{4 \times 4} \end{pmatrix} = -\hat{C}^{-1}. \tag{21}$$

To obtain the explicit form of the charge conjugation matrix, we have to fix a representation of the gamma matrices. We thus found the matrix (21) in the Pauli representation. Explicit forms of wave functions of the form (19) to (20) are shown in the appendix D.1.

4 An 8-dimensional realization of the Clifford algebra in the 5-dimensional Galilean space-time

In this section, we turn to the reduction from the $(5+1)$ Minkowski manifold to the $(4+1)$ Galilean space-time. More specifically, we exploit the results found in the previous sections to obtain 8×8 gamma matrices (denoted Γ) in the Galilean space-time, from the gamma matrices (denoted γ) defined on the extended Minkowski manifold.

Consider the 5-dimensional Galilean space-time with light-cone coordinates, x^μ ($\mu = 1, \dots, 5$), with the metric tensor:

$$\eta_{\mu\nu} = \begin{pmatrix} \mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 2} \\ \mathbf{0}_{2 \times 3} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

The coordinate system y^μ ($\mu = 1, 2, 3, 4, 0$), defined by (see Ref. [2])

$$\mathbf{y} = \mathbf{x}, \quad y^4 = \frac{1}{\sqrt{2}}(x^4 - x^5), \quad y^0 = \frac{1}{\sqrt{2}}(x^4 + x^5), \tag{22}$$

admits the diagonal metric of Eq. (2). Therefore, the 5-dimensional Galilean space-time corresponds to a $(4+1)$ Minkowski space-time, so that it is possible to describe

non-relativistic theories in a Lorentz-like covariant form. A further reduction, to the Newtonian space-time, is needed, as explained in Refs. [1, 2].

In order to introduce pseudo-tensor interactions of rank 0, 1 and 2 into the 5-dimensional Galilean theory, we need a gamma-6 matrix (which corresponds to the gamma-5 matrix in the usual $(3+1)$ Minkowski space-time) obtained by dimensional reduction from the $(5+1)$ Minkowski space-time to the $(4+1)$ Minkowski space-time with light-cone coordinates.

Let Γ^μ and γ^μ be 8×8 gamma matrices in the 5-dimensional Galilean and Minkowski space-times, respectively. They transform as the contravariant vectors in each space-time. Therefore, we have

$$\begin{aligned}\mathbf{\Gamma} &= \gamma, \\ \Gamma^4 &= \frac{1}{\sqrt{2}}(\gamma^4 + \gamma^0), \\ \Gamma^5 &= \frac{1}{\sqrt{2}}(-\gamma^4 + \gamma^0).\end{aligned}\tag{23}$$

The gamma-6 matrix may be taken as

$$\Gamma^6 = \gamma^7,$$

where Γ^6 anticommutes with Γ^μ . Note that neither $\gamma^1\gamma^2\gamma^3\gamma^4\gamma^0$ nor $\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5$ anticommute with the Γ^μ s, which satisfy the Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}.$$

The parity matrix Π may be expressed by

$$\Pi = \gamma^4\gamma^5\gamma^0,$$

and satisfies the relations

$$\begin{aligned}\Pi\Gamma^k + \Gamma^k\Pi &= 0, & (k = 1, 2, 3), \\ \Pi\Gamma^4 - \Gamma^4\Pi &= 0, \\ \Pi\Gamma^5 - \Gamma^5\Pi &= 0.\end{aligned}$$

These equations are equivalent to imposing the condition given by Eq. (10).

Since, in the 5-dimensional Galilean space-time, the dimension of algebra is $2^5 = 32$, then we take thirty-two independent gamma matrices given by

$$\Gamma_A = \mathbb{I}, \Gamma^6, \Gamma_\mu, i\Gamma^6\Gamma_\mu, \Sigma_{\mu\nu}, \Gamma^6\Sigma_{\mu\nu},$$

where $\Sigma_{\mu\nu}$ is defined by

$$\Sigma_{\mu\nu} = \frac{1}{2i}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu).\tag{24}$$

These Γ -matrices satisfy the relation:

$$\text{Tr}(\Gamma^A\Gamma_B) = 8\delta^A_B.\tag{25}$$

Since the Γ^μ s are linear combinations of γ^μ s and $\Gamma^6 = \gamma^7$, we have

$$(\Gamma^\mu)^T = -C^{-1}\Gamma^\mu C = \hat{C}^{-1}(\Gamma^\mu)^\dagger \hat{C}.$$

Thus we find

$$(\Gamma_A C)_{\alpha\beta} = \epsilon_A (\Gamma_A C)_{\beta\alpha}, \quad (26)$$

where

$$\epsilon_A = \begin{cases} +1 & \text{for } C, \Gamma^6 \Sigma_{\mu\nu} C, \\ -1 & \text{for } \Gamma^6 C, \Gamma_\mu C, i\Gamma^6 \Gamma_\mu C, \Sigma_{\mu\nu} C. \end{cases} \quad (27)$$

4.1 The Dirac-type equation in the Pauli representation

In the 5-dimensional Galilean space-time, the Dirac-type equation for massless fields can be cast in the following form:

$$\Lambda(\partial)\psi(x) = 0, \quad (28)$$

with

$$\Lambda(\partial) = -\Gamma^\mu \partial_\mu,$$

where the wave function is an 8-component spinor. The adjoint equation to Eq. (28) is given by

$$\bar{\psi}(x)\Lambda(-\overleftarrow{\partial}) = 0,$$

and

$$\bar{\psi}(x) = \psi^\dagger(x)\eta.$$

Also, we use

$$\eta = i\frac{1}{\sqrt{2}}(\Gamma^4 + \Gamma^5) = i\gamma^0, \quad (29)$$

which agrees with Eq. (11). Here, we have imposed the relation:

$$[\eta\Lambda(\partial)]^\dagger = \eta\Lambda(-\partial).$$

For the fifth component of the derivative ∂_μ , we have the relationship $\partial_5 = -im$, which implies the ansatz

$$\psi(x) = e^{-imx^5}\psi(\mathbf{x}, t),$$

or, in the matrix form,

$$\begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ u_4(x) \end{pmatrix} = e^{-imx^5} \begin{pmatrix} u_1(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) \\ u_3(\mathbf{x}, t) \\ u_4(\mathbf{x}, t) \end{pmatrix}$$

where $u_k(x)$ and $u_k(\mathbf{x}, t)$ ($k = 1, 2, 3, 4$) are two-component spinors.

The Galilean Γ -matrices can be expressed in terms of the γ -matrices in the $(5+1)$ Minkowski space-time. By using Eqs. (7), we obtain the Dirac-type equation in the Pauli representation. If we write it out explicitly, we have

$$i\partial_0[u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t)] = -\frac{1}{2m}\Delta[u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t)], \quad (30)$$

$$i\partial_0[u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t)] = -\frac{1}{2m}\Delta[u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t)],$$

with

$$\begin{aligned} u_1(\mathbf{x}, t) - u_3(\mathbf{x}, t) &= \frac{1}{\sqrt{2}m}\sigma \cdot \nabla[u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t)] , \\ u_2(\mathbf{x}, t) + u_4(\mathbf{x}, t) &= \frac{1}{\sqrt{2}m}\sigma \cdot \nabla[u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t)] . \end{aligned} \quad (31)$$

It is convenient to introduce the orthogonal matrix R :

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & I & 0 \\ 0 & I & 0 & I \\ I & 0 & -I & 0 \\ 0 & I & 0 & -I \end{pmatrix} = \frac{1}{\sqrt{2}}(\rho_1 + \rho_3). \quad (32)$$

We can utilize this matrix to rotate $\psi(\mathbf{x}, t)$ in the form:

$$\Psi(\mathbf{x}, t) = R\psi(\mathbf{x}, t).$$

Written explicitly in matrix form, it reads

$$\begin{pmatrix} U_1(\mathbf{x}, t) \\ U_2(\mathbf{x}, t) \\ U_3(\mathbf{x}, t) \\ U_4(\mathbf{x}, t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) + u_4(\mathbf{x}, t) \\ u_1(\mathbf{x}, t) - u_3(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t) \end{pmatrix}.$$

Therefore, we obtain from Eqs. (30) to (31) that

$$i\partial_0 U_1(\mathbf{x}, t) = -\frac{1}{2m}\Delta U_1(\mathbf{x}, t) , \quad (33)$$

$$U_2(\mathbf{x}, t) = \frac{1}{\sqrt{2}m}\sigma \cdot \nabla U_1(\mathbf{x}, t) ,$$

$$U_3(\mathbf{x}, t) = \frac{1}{\sqrt{2}m}\sigma \cdot \nabla U_4(\mathbf{x}, t) ,$$

$$i\partial_0 U_4(\mathbf{x}, t) = -\frac{1}{2m}\Delta U_4(\mathbf{x}, t) . \quad (34)$$

This result shows that the 5-dimensional Galilean matrices can be obtained by using a similarity transformation which involves the orthogonal matrix R .

4.2 Explicit forms of the Galilean gamma matrices

Consider the Dirac Lagrangian, written as

$$\mathcal{L}(x) = \bar{\psi}(x)\Lambda(x)\psi(x),$$

where

$$\Lambda(\partial) = -\Gamma^\mu \partial_\mu.$$

The hermiticity of the Lagrangian leads to the condition given by Eq. (13). This Lagrangian becomes

$$\mathcal{L}(x) = \bar{\Psi}(x)\tilde{\Lambda}(\partial)\Psi(x), \quad (35)$$

where Ψ is given by

$$\Psi(x) = e^{-imx^5}\Psi(\mathbf{x}, t) = e^{-imx^5}R\psi(\mathbf{x}, t),$$

and $\tilde{\Lambda}$ is defined as

$$\tilde{\Lambda}(\partial) = R\Lambda(\partial)R^{-1}. \quad (36)$$

Note that

$$R = R^T = R^{-1},$$

Therefore, it follows from Eq. (36) that

$$\tilde{\Gamma}^\mu = R\Gamma^\mu R^{-1},$$

$$\tilde{\eta} = R\eta R^{-1}.$$

The Dirac-type equation is obtained from the Lagrangian given by Eq. (35):

$$\tilde{\Lambda}(\partial)\Psi(\mathbf{x}, t) = 0. \quad (37)$$

If we express the Galilean gamma matrices in the Pauli representation, then the Dirac-type equation (37) leads to Eqs. (33) to (34).

Explicit forms of the Galilean gamma matrices are given by using the Pauli representation as follows:

$$\Gamma^k = \begin{pmatrix} \mathbf{0}_{4 \times 4} & 0 & \sigma_k \\ 0 & -\sigma_k & 0 \\ \sigma_k & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix}, \quad (k = 1, 2, 3),$$

$$\Gamma^4 = -\sqrt{2}i \begin{pmatrix} \mathbf{0}_{4 \times 4} & 0 & 0 \\ I & 0 & I \\ 0 & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix},$$

$$\Gamma^5 = -\sqrt{2}i \begin{pmatrix} & I & 0 \\ \mathbf{0}_{4 \times 4} & 0 & 0 \\ 0 & 0 & \\ 0 & I & \mathbf{0}_{4 \times 4} \end{pmatrix},$$

$$\eta = \begin{pmatrix} & I & 0 \\ \mathbf{0}_{4 \times 4} & 0 & I \\ I & 0 & \\ 0 & I & \mathbf{0}_{4 \times 4} \end{pmatrix} = \rho_1.$$

Moreover, we find

$$\Gamma^6 = \begin{pmatrix} -I & 0 & \\ 0 & -I & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & I & 0 \\ & 0 & I \end{pmatrix} = -\rho_3,$$

$$\Pi = \begin{pmatrix} & 0 & -iI \\ \mathbf{0}_{4 \times 4} & iI & 0 \\ 0 & -iI & \\ iI & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix}.$$

Here we have replaced $\tilde{\Gamma}^\mu, \tilde{\Gamma}^6, \tilde{\eta}$ and $\tilde{\Pi}$ by $\Gamma^\mu, \Gamma^6, \eta$ and Π , respectively. It should be noted that the matrix Γ^6 is block diagonal.

The chirality operators may be defined as

$$\frac{1}{\sqrt{2}}(\mathbb{I} \pm \Gamma^6).$$

They are block diagonal, and the chiral eigenstates are given by

$$\psi_\pm(x) = \frac{1}{\sqrt{2}}(\mathbb{I} \mp \Gamma^6)\psi(x).$$

5 Construction of wave functions for the Dirac-type equation

The main advantage of employing a 5-dimensional Galilean covariant theory is that we can perform many calculations in a way analogous to the relativistic treatment. Indeed, many of our non-relativistic equations have the same form as the corresponding equations in relativistic quantum theory, except that they are written in a manifestly covariant form on the $(4 + 1)$ Minkowski space-time.

Let P^μ and p^μ be contravariant vectors in the 5-dimensional Galilean and Minkowski space-times, respectively. Then they are written as

$$P^\mu = (\mathbf{p}, m, E), \quad (38)$$

$$p^\mu = (\mathbf{p}, p^4, p^0),$$

with

$$p^0 = \frac{1}{\sqrt{2}}(m + E), \quad p^4 = \frac{1}{\sqrt{2}}(m - E),$$

where we have used Eq. (22). Moreover, if we impose the conditions

$$P_\mu P^\mu = p_\mu p^\mu = -\kappa_m^2,$$

and

$$\kappa_m = \sqrt{2}m.$$

we find

$$E = \frac{1}{2m}\mathbf{p} \cdot \mathbf{p} + m.$$

We find a similar expression for p^0 :

$$p^0 = \sqrt{\mathbf{p} \cdot \mathbf{p} + (p^4)^2 + \kappa_m^2} = \frac{1}{2\kappa_m}\mathbf{p} \cdot \mathbf{p} + \kappa_m.$$

When we perform the reduction from the 6-dimensional to the 5-dimensional Minkowski space-time, the Lorentz boost, Eq. (14), becomes

$$L(\mathbf{p}, p^4) = \sqrt{\frac{p^0 + \kappa_m}{2\kappa_m}} \mathbb{I} - \frac{1}{\sqrt{2\kappa_m(p^0 + \kappa_m)}} \gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + p^4 \gamma^4) = L^{-1}(-\mathbf{p}, -p^4). \quad (39)$$

The Galilean transformation matrix is obtained by substituting Eq. (23) into Eq. (39):

$$L(\mathbf{p}, p^4) = \frac{1}{\sqrt{2\kappa_m(p^0 + \kappa_m)}} \left[(p^0 + \kappa_m) \mathbb{I} - \frac{1}{\sqrt{2}} (\Gamma^4 + \Gamma^5) \mathbf{p} \cdot \boldsymbol{\Gamma} - \frac{1}{2} (-\Gamma^4 \Gamma^5 + \Gamma^5 \Gamma^4) p^4 \right] =: G(P). \quad (40)$$

Hence we find

$$G^{-1}(P) \Gamma^\mu G(P) = Z^\mu_\nu(P) \Gamma^\nu,$$

where

$$Z^i_\nu(P) = \left(\eta^{ik} + \frac{P^i P^k}{m(E + 3m)}, \frac{2P^i}{E + 3m}, \frac{(E + m)P^i}{m(E + 3m)} \right),$$

$$Z^4_\nu(P) = \left(\frac{2P^k}{E + 3m}, \frac{4m}{E + 3m}, \frac{E - m}{E + 3m} \right),$$

$$Z^5_\nu(P) = \left(\frac{(E+m)P^k}{m(E+3m)}, \frac{E-m}{E+3m}, \frac{(E+m)^2}{m(E+3m)} \right).$$

The transformation coefficients Z^μ_ν lead to

$$\eta_{\mu\nu} Z^\mu_\rho(P) Z^\nu_\sigma(P) = \eta_{\rho\sigma}.$$

By noticing that

$$d(ip) = -(i\gamma \cdot p - \kappa_m) = -(i\Gamma \cdot P - \kappa_m) =: D(iP),$$

we find

$$G^{-1}(P) D(iP) G(P) = \kappa_m \left[\mathbb{I} + i \frac{1}{\sqrt{2}} (\Gamma^4 + \Gamma^5) \right]. \quad (41)$$

We can prove the following relations:

$$G(P)(\mathbb{I} + \eta) = \frac{1}{\sqrt{2m(E+3m)}} D(iP) (\mathbb{I} + \eta),$$

$$(\mathbb{I} + \eta) G^\dagger(P) \eta = (\mathbb{I} + \eta) G^{-1}(P),$$

where we have used

$$\eta G^\dagger(P) \eta = G^{-1}(P),$$

with η defined by Eq. (29). If we choose the Pauli representation for gamma matrices, then Eq. (41) shows us that the Klein-Gordon divisor in the 5-dimensional Galilean space-time is diagonalized by the Galilean boost, Eq. (40).

Following the prescription developed in section 3, we can construct wave functions for the Dirac-type equation:

$$-(i\gamma \cdot p + \kappa_m) u^{(r)}(\mathbf{p}, p^4) = -(i\Gamma \cdot P + \kappa_m) u^{(r)}(P) = 0,$$

where the matrices Γ^μ are given by Eq. (23), in the Pauli representation. The wave functions then take the form:

$$\begin{aligned} h_\alpha^\beta u_\beta^{(r)}(P) &= \frac{1}{2} \epsilon^{(r)} u_\alpha^{(r)}(P), \\ &= \frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[(-i\Gamma \cdot P + \kappa_m) \frac{1}{2} (\mathbb{I} + \eta) S(\mathbf{P}) \Sigma_3 \right]_\alpha^r, \end{aligned} \quad (42)$$

$$\begin{aligned} \bar{u}^{(r)\beta}(P) h_\beta^\alpha &= \frac{1}{2} \epsilon^{(r)} \bar{u}^{(r)\alpha}(P), \\ &= \frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[\Sigma_3 S^{-1}(\mathbf{P}) \frac{1}{2} (\mathbb{I} + \eta) (-i\Gamma \cdot P + \kappa_m) \right]_r^\alpha, \end{aligned}$$

$$\begin{aligned} h_\alpha^\beta v_\beta^{(r)}(P) &= -\frac{1}{2} \epsilon^{(r)} v_\alpha^{(r)}(P), \\ &= \frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[(i\Gamma \cdot P + \kappa_m) \frac{1}{2} (\mathbb{I} - \eta) S(\mathbf{P}) \Sigma_3 \hat{C} \right]_{\alpha r}, \end{aligned}$$

$$\begin{aligned} \bar{v}^{(r)\beta}(P) h_\beta^\alpha &= -\frac{1}{2} \epsilon^{(r)} \bar{v}^{(r)\alpha}(P), \\ &= -\frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[\hat{C}^{-1} \Sigma_3 S(\mathbf{P}) \frac{1}{2} (\mathbb{I} - \eta) (i\Gamma \cdot P + \kappa_m) \right]^{r\alpha}, \end{aligned} \quad (43)$$

where the charge conjugation matrix \hat{C} is given by Eq. (21). These wave functions are given explicitly in appendix D.2.

6 Concluding remarks

The general idea allowing a covariant treatment of non-relativistic theories is to perform a dimensional reduction from $(4 + 1)$ Minkowski space-time. However, this prevents the existence of parity operator, since the γ^5 -like matrix has no analogue in odd-dimensional space-time. Therefore, in this article, we start with a $(5 + 1)$ space-time.

An 8-dimensional realization of the Clifford algebra in the 5-dimensional Galilean space-time is obtained by reduction from the 6-dimensional to the 5-dimensional Minkowski space-time which encompasses Galilean space-time. The solutions to the Dirac-type equation in the 5-dimensional Galilean space-time are shown explicitly in the Pauli representation (see appendix D.2). The chiral eigenstates are also obtained by rotating the solution just mentioned above by means of Eq. (32).

Consider an inverse Galilean transformation, obtained by substituting the direction (\mathbf{p}, p^4) with $(-\mathbf{p}, -p^4)$. Then we can derive the Galilean boost from the Lorentz boost, Eq. (39),

$$L^{-1}(-\mathbf{p}, -p^4) = L(\mathbf{p}, p^4) = G(P) ,$$

and hence

$$G(P)\Gamma^\mu G^{-1}(P) = \Gamma'^\mu Z_\nu{}^\mu(P),$$

where

$$\Gamma'^\mu = (-\Gamma, \Gamma^5, \Gamma^4).$$

It should be mentioned that Γ^4 and Γ^5 are interchanged by substituting p^4 with $-p^4$. Thus, in the massless limit, we find

$$\lim_{m \rightarrow 0} Z_\nu{}^\mu(P) = \begin{pmatrix} 1 & 0 & 0 & v^1 & 0 \\ 0 & 1 & 0 & v^2 & 0 \\ 0 & 0 & 1 & v^3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v^1 & v^2 & v^3 & \frac{1}{2}\mathbf{v} \cdot \mathbf{v} & 1 \end{pmatrix} ,$$

which is exactly the proper Galilean transformation.

The construction presented in this article allows a definition of a parity operator as well as a chirality operator. We have included important information related to the Clifford algebra, such as the commutation and anticommutation relations, trace formulas and the Fierz identities, in the appendices A, B and C, respectively. A reduction to the $(4 + 1)$ Minkowski space-time that encompasses the Galilean space-time is deduced.

Now the necessary developments to treat Galilean covariant theories for applications to problems like β -decay and to develop a theory like the Galilean version of the Nambu-Jona-Lasinio problem are possible.

A Commutation and anticommutation relations for the gamma matrices

Hereafter, we provide lists of commutation and anticommutation relations for the gamma matrices. Section A.1 contains these relations for the 8×8 representations discussed in section 2.1, for the $(5 + 1)$ Minkowski space-time. The corresponding relations for the 5-dimensional Galilean space-time Γ -matrices, introduced in section 4, are given in section A.2.

A.1 The $(5 + 1)$ Minkowski space-time

The quantities encountered hereafter are defined in section 2.1. The matrices $\Sigma^{[\]}$ are described at the end of the present section.

$$\begin{aligned}
[\gamma^7, \gamma^\mu] &= 2\gamma^7 \gamma^\mu, \\
[\gamma^7, i\gamma^7 \gamma^\mu] &= 2i\gamma^\mu, \\
[\gamma^7, \sigma^{\mu\nu}] &= 0, \\
[\gamma^7, \gamma^7 \sigma^{\mu\nu}] &= 0, \\
[\gamma^7, \sigma^{\lambda\mu\nu}] &= 2\gamma^7 \sigma^{\lambda\mu\nu}, \\
[\gamma^\mu, \gamma^\nu] &= 2i\sigma^{\mu\nu}, \\
[\gamma^\mu, i\gamma^7 \gamma^\nu] &= -i\gamma^7 \{\gamma^\mu, \gamma^\nu\}, \\
[\gamma^\rho, \sigma^{\mu\nu}] &= -2i(g^{\rho\mu}\gamma^\nu - g^{\rho\nu}\gamma^\mu), \\
[\gamma^\lambda, \gamma^7 \sigma^{\mu\nu}] &= -\gamma^7 \{\gamma^\lambda, \sigma^{\mu\nu}\}, \\
[\gamma^\kappa, \sigma^{\lambda\mu\nu}] &= -\frac{1}{6}\epsilon^{\lambda\mu\nu\rho\sigma\tau}\gamma^7 \{\gamma^\kappa, \sigma_{\rho\sigma\tau}\}, \\
[i\gamma^7 \gamma^\mu, i\gamma^7 \gamma^\nu] &= [\gamma^\mu, \gamma^\nu], \\
[i\gamma^7 \gamma^\rho, \sigma^{\mu\nu}] &= i\gamma^7 [\gamma^\rho, \sigma^{\mu\nu}], \\
[i\gamma^7 \gamma^\lambda, \gamma^7 \sigma^{\mu\nu}] &= i\gamma^7 \{\gamma^\lambda, \sigma^{\mu\nu}\}, \\
[i\gamma^7 \gamma^\rho, \sigma^{\lambda\mu\nu}] &= i\gamma^7 \{\gamma^\rho, \sigma^{\lambda\mu\nu}\}, \\
[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] &= 2i(g^{\mu\sigma}\sigma^{\rho\nu} - g^{\mu\rho}\sigma^{\sigma\nu} - g^{\nu\sigma}\sigma^{\rho\mu} + g^{\nu\rho}\sigma^{\sigma\mu}), \\
[\sigma^{\mu\nu}, \gamma^7 \sigma^{\rho\sigma}] &= \gamma^7 [\sigma^{\mu\nu}, \sigma^{\rho\sigma}], \\
[\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}] &= 2i[(g^{\lambda\rho}\sigma^{\sigma\mu\nu} - g^{\lambda\sigma}\sigma^{\rho\mu\nu}) + (g^{\mu\rho}\sigma^{\sigma\nu\lambda} - g^{\mu\sigma}\sigma^{\rho\nu\lambda}) + \\
&\quad + (g^{\nu\rho}\sigma^{\sigma\lambda\mu} - g^{\nu\sigma}\sigma^{\rho\lambda\mu})], \\
&= -\frac{1}{6}\epsilon^{\lambda\mu\nu\kappa\tau\eta}[\sigma^{\rho\sigma}, \sigma_{\kappa\tau\eta}] \gamma^7, \\
&= -i\epsilon^{\lambda\mu\nu\kappa\tau\eta}(g^\rho_\kappa \sigma^\sigma_{\tau\eta} - g^\sigma_\kappa \sigma^\rho_{\tau\eta})\gamma^7,
\end{aligned}$$

$$\begin{aligned}
[\gamma^7 \sigma^{\mu\nu}, \gamma^7 \sigma^{\rho\sigma}] &= [\sigma^{\mu\nu}, \sigma^{\rho\sigma}], \\
[\gamma^7 \sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}] &= \gamma^7 \{\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}\}, \\
[\sigma^{\lambda\mu\nu}, \sigma^{\rho\sigma\tau}] &= 2(i\Sigma^{[\lambda\mu\nu][\rho\sigma\tau]} + \epsilon^{\lambda\mu\nu\rho\sigma\tau} \gamma^7), \\
\{\gamma^7, \gamma^\mu\} &= 0, \\
\{\gamma^7, i\gamma^7 \gamma^\mu\} &= 0, \\
\{\gamma^7, \sigma^{\mu\nu}\} &= 2\gamma^7 \sigma^{\mu\nu}, \\
\{\gamma^7, \gamma^7 \sigma^{\mu\nu}\} &= 2\sigma^{\mu\nu}, \\
\{\gamma^7, \sigma^{\lambda\mu\nu}\} &= 0, \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}, \\
\{\gamma^\mu, i\gamma^7 \gamma^\nu\} &= -i\gamma^7 [\gamma^\mu, \gamma^\nu], \\
\{\gamma^\lambda, \sigma^{\mu\nu}\} &= -\frac{1}{3} \epsilon^{\lambda\mu\nu\rho\sigma\tau} \sigma_{\rho\sigma\tau} \gamma^7 = 2\sigma^{\lambda\mu\nu}, \\
\{\gamma^\rho, \gamma^7 \sigma^{\mu\nu}\} &= -\gamma^7 [\gamma^\rho, \sigma^{\mu\nu}], \\
\{\gamma^\rho, \sigma^{\lambda\mu\nu}\} &= 2(g^{\rho\lambda} \sigma^{\mu\nu} + g^{\rho\mu} \sigma^{\nu\lambda} + g^{\rho\nu} \sigma^{\lambda\mu}), \\
\{i\gamma^7 \gamma^\mu, i\gamma^7 \gamma^\nu\} &= \{\gamma^\mu, \gamma^\nu\}, \\
\{i\gamma^7 \gamma^\lambda, \sigma^{\mu\nu}\} &= i\gamma^7 \{\gamma^\lambda, \sigma^{\mu\nu}\}, \\
\{i\gamma^7 \gamma^\rho, \gamma^7 \sigma^{\mu\nu}\} &= -i[\gamma^\rho, \sigma^{\mu\nu}], \\
\{i\gamma^7 \gamma^\kappa, \sigma^{\lambda\mu\nu}\} &= i\gamma^7 [\gamma^\kappa, \sigma^{\lambda\mu\nu}], \\
\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\} &= 2(\Sigma^{[\mu\nu][\rho\sigma]} + i\epsilon^{\mu\nu\rho\sigma\eta\xi} \sigma_{\eta\xi} \gamma^7), \\
\{\sigma^{\mu\nu}, \gamma^7 \sigma^{\rho\sigma}\} &= \gamma^7 \{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\}, \\
\{\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}\} &= 2(\Sigma^{[\rho\sigma][\lambda\mu\nu]} - \epsilon^{\rho\sigma\lambda\mu\nu\eta} \gamma_\eta \gamma^7), \\
\{\gamma^7 \sigma^{\mu\nu}, \gamma^7 \sigma^{\rho\sigma}\} &= \{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\}, \\
\{\gamma^7 \sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}\} &= \gamma^7 [\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}], \\
\{\sigma^{\lambda\mu\nu}, \sigma^{\rho\sigma\tau}\} &= \frac{1}{3} [2(g^{\lambda\rho} \sigma^{\sigma\tau} + g^{\lambda\sigma} \sigma^{\tau\rho} + g^{\lambda\tau} \sigma^{\rho\sigma}) \sigma^{\mu\nu} + \\
&\quad + 2(g^{\mu\rho} \sigma^{\sigma\tau} + g^{\mu\sigma} \sigma^{\tau\rho} + g^{\mu\tau} \sigma^{\rho\sigma}) \sigma^{\nu\lambda} + \\
&\quad + 2(g^{\nu\rho} \sigma^{\sigma\tau} + g^{\nu\sigma} \sigma^{\tau\rho} + g^{\nu\tau} \sigma^{\rho\sigma}) \sigma^{\lambda\mu}] \\
&\quad - \frac{1}{3} i [\gamma^7 \gamma^\lambda (\epsilon^{\rho\sigma\tau\mu\zeta\xi} \sigma_{\zeta\xi}^\nu - \epsilon^{\rho\sigma\tau\nu\zeta\xi} \sigma_{\zeta\xi}^\mu) + \\
&\quad + \gamma^7 \gamma^\mu (\epsilon^{\rho\sigma\tau\nu\zeta\xi} \sigma_{\zeta\xi}^\lambda - \epsilon^{\rho\sigma\tau\lambda\zeta\xi} \sigma_{\zeta\xi}^\nu) + \\
&\quad + \gamma^7 \gamma^\nu (\epsilon^{\rho\sigma\tau\lambda\zeta\xi} \sigma_{\zeta\xi}^\mu - \epsilon^{\rho\sigma\tau\mu\zeta\xi} \sigma_{\zeta\xi}^\lambda)].
\end{aligned}$$

The matrices $\Sigma^{[\] []}$ are defined by:

$$\begin{aligned}
\Sigma^{[\mu\nu][\rho\sigma]} &= g^{\mu\rho} g^{\sigma\nu} - g^{\mu\sigma} g^{\rho\nu}, \\
\Sigma^{[\rho\sigma][\lambda\mu\nu]} &= (g^{\rho\lambda} g^{\sigma\mu} - g^{\sigma\lambda} g^{\rho\mu}) \gamma^\nu + (g^{\rho\mu} g^{\sigma\nu} - g^{\sigma\mu} g^{\rho\nu}) \gamma^\lambda + (g^{\rho\nu} g^{\sigma\lambda} - g^{\sigma\nu} g^{\rho\lambda}) \gamma^\mu, \\
\Sigma^{[\lambda\mu\nu][\rho\sigma\tau]} &= (g^{\lambda\rho} g^{\mu\sigma} - g^{\lambda\sigma} g^{\mu\rho}) \sigma^{\nu\tau} + (g^{\lambda\sigma} g^{\mu\tau} - g^{\lambda\tau} g^{\mu\sigma}) \sigma^{\nu\rho} + (g^{\lambda\tau} g^{\mu\rho} - g^{\lambda\rho} g^{\mu\tau}) \sigma^{\nu\sigma} + \\
&\quad + (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \sigma^{\lambda\tau} + (g^{\mu\sigma} g^{\nu\tau} - g^{\mu\tau} g^{\nu\sigma}) \sigma^{\lambda\rho} + (g^{\mu\tau} g^{\nu\rho} - g^{\mu\rho} g^{\nu\tau}) \sigma^{\lambda\sigma} + \\
&\quad + (g^{\nu\rho} g^{\lambda\sigma} - g^{\nu\sigma} g^{\lambda\rho}) \sigma^{\mu\tau} + (g^{\nu\sigma} g^{\lambda\tau} - g^{\nu\tau} g^{\lambda\sigma}) \sigma^{\mu\rho} + (g^{\nu\tau} g^{\lambda\rho} - g^{\nu\rho} g^{\lambda\tau}) \sigma^{\mu\sigma}.
\end{aligned}$$

A.2 The 5-dimensional Galilean space-time

The quantities described in this appendix are described in section 4. The matrices $\Sigma^{\mu\nu}$ are defined in Eq. (24).

$$\begin{aligned}
[\Gamma^6, \Gamma^\mu] &= 2\Gamma^6\Gamma^\mu, \\
[\Gamma^6, i\Gamma^6\Gamma^\mu] &= 2i\Gamma^\mu, \\
[\Gamma^6, \Sigma^{\mu\nu}] &= 0, \\
[\Gamma^6, \Gamma^6\Sigma^{\mu\nu}] &= 0, \\
[\Gamma^\mu, \Gamma^\nu] &= 2i\Sigma^{\mu\nu}, \\
[\Gamma^\mu, i\Gamma^6\Gamma^\nu] &= -i\Gamma^6\{\Gamma^\mu, \Gamma^\nu\}, \\
[\Gamma^\rho, \Sigma^{\mu\nu}] &= -2i(\eta^{\rho\mu}\Gamma^\nu - \eta^{\rho\nu}\Gamma^\mu), \\
[\Gamma^\lambda, \Gamma^6\Sigma^{\mu\nu}] &= -\Gamma^6\{\Gamma^\lambda, \Sigma^{\mu\nu}\}, \\
[i\Gamma^6\Gamma^\mu, i\Gamma^6\Gamma^\nu] &= [\Gamma^\mu, \Gamma^\nu], \\
[i\Gamma^6\Gamma^\rho, \Sigma^{\mu\nu}] &= i\Gamma^6[\Gamma^\rho, \Sigma^{\mu\nu}], \\
[i\Gamma^6\Gamma^\lambda, \Gamma^6\Sigma^{\mu\nu}] &= -i\{\Gamma^\lambda, \Sigma^{\mu\nu}\}, \\
[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] &= 2i(\eta^{\mu\sigma}\Sigma^{\rho\nu} - \eta^{\mu\rho}\Sigma^{\sigma\nu} - \eta^{\nu\sigma}\Sigma^{\rho\mu} + \eta^{\nu\rho}\Sigma^{\sigma\mu}), \\
[\Sigma^{\mu\nu}, \Gamma^6\Sigma^{\rho\sigma}] &= \Gamma^6[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}], \\
[\Gamma^6\Sigma^{\mu\nu}, \Gamma^6\Sigma^{\rho\sigma}] &= [\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}], \\
\{\Gamma^6, \Gamma^\mu\} &= 0, \\
\{\Gamma^6, i\Gamma^6\Gamma^\mu\} &= 0, \\
\{\Gamma^6, \Sigma^{\mu\nu}\} &= 2\Gamma^6\Sigma^{\mu\nu}, \\
\{\Gamma^6, \Gamma^6\Sigma^{\mu\nu}\} &= 2\Sigma^{\mu\nu}, \\
\{\Gamma^\mu, \Gamma^\nu\} &= 2\eta^{\mu\nu}, \\
\{\Gamma^\mu, i\Gamma^6\Gamma^\nu\} &= -i\Gamma^6[\Gamma^\mu, \Gamma^\nu], \\
\{\Gamma^\lambda, \Sigma^{\mu\nu}\} &= i(\Gamma^\lambda\Gamma^\nu\Gamma^\mu - \Gamma^\mu\Gamma^\nu\Gamma^\lambda), \\
\{\Gamma^\rho, \Gamma^6\Sigma^{\mu\nu}\} &= -\Gamma^6[\Gamma^\rho, \Sigma^{\mu\nu}], \\
\{i\Gamma^6\Gamma^\mu, i\Gamma^6\Gamma^\nu\} &= \{\Gamma^\mu, \Gamma^\nu\}, \\
\{i\Gamma^6\Gamma^\lambda, \Sigma^{\mu\nu}\} &= i\Gamma^6\{\Gamma^\lambda, \Sigma^{\mu\nu}\}, \\
\{i\Gamma^6\Gamma^\rho, \Gamma^6\Sigma^{\mu\nu}\} &= -i[\Gamma^\rho, \Sigma^{\mu\nu}], \\
\{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\} &= (\Gamma^\mu\Gamma^\nu\Gamma^\sigma\Gamma^\rho + \Gamma^\rho\Gamma^\sigma\Gamma^\nu\Gamma^\mu) - 2\eta^{\mu\nu}\eta^{\rho\sigma}, \\
\{\Sigma^{\mu\nu}, \Gamma^6\Sigma^{\rho\sigma}\} &= \Gamma^6\{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\}, \\
\{\Gamma^6\Sigma^{\mu\nu}, \Gamma^6\Sigma^{\rho\sigma}\} &= \{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\}.
\end{aligned}$$

B Traces of the gamma matrices

In this appendix, we give lists of traces involving the gamma matrices. The γ -matrices defined in section 2.1 for the $(5 + 1)$ Minkowski space-time are given in section B.1, and the Γ -matrices of section 4 for the 5-dimensional Galilean space-time are in B.2.

B.1 The $(5 + 1)$ Minkowski space-time

$$\begin{aligned}
\text{Tr}(\gamma_{\mu_1} \cdots \gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\gamma_\mu \gamma_\nu) &= 8 g_{\mu\nu}, \\
\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) &= 8 (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}), \\
\text{Tr}(\gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau) &= 8 [(g_{\lambda\mu} g_{\nu\rho} - g_{\lambda\nu} g_{\mu\rho} + g_{\lambda\rho} g_{\mu\nu}) g_{\sigma\tau} - g_{\lambda\rho} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}) \\
&\quad - (g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\nu} g_{\mu\sigma} + g_{\lambda\sigma} g_{\mu\nu}) g_{\tau\rho} - g_{\lambda\sigma} (g_{\mu\tau} g_{\nu\rho} - g_{\mu\rho} g_{\nu\tau}) \\
&\quad + (g_{\lambda\mu} g_{\nu\tau} - g_{\lambda\nu} g_{\mu\tau} + g_{\lambda\tau} g_{\mu\nu}) g_{\rho\sigma} - g_{\lambda\tau} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})], \\
\text{Tr}(\gamma^7) &= 0, \\
\text{Tr}(\gamma^7 \gamma_{\mu_1} \cdots \gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\gamma^7 \gamma_\mu \gamma_\nu) &= 0, \\
\text{Tr}(\gamma^7 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) &= 0, \\
\text{Tr}(\gamma^7 \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau) &= -8 \epsilon_{\lambda\mu\nu\rho\sigma\tau}, \\
\text{Tr}(\sigma_{\mu\nu}) &= 0, \\
\text{Tr}(\sigma_{\lambda\mu\nu}) &= 0, \\
\text{Tr}(\sigma_{\mu\nu} \sigma_{\rho\sigma}) &= 8 (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \\
\text{Tr}(\sigma_{\lambda\mu} \sigma_{\nu\rho} \sigma_{\sigma\tau}) &= 8i [(g_{\lambda\tau} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\tau}) g_{\rho\sigma} - (g_{\lambda\sigma} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\sigma}) g_{\tau\rho} \\
&\quad - g_{\lambda\rho} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}) + g_{\mu\rho} (g_{\nu\sigma} g_{\lambda\tau} - g_{\nu\tau} g_{\lambda\sigma})], \\
\text{Tr}(\sigma_{\lambda\mu\nu} \sigma_{\rho\sigma\tau}) &= \frac{8}{3} [g_{\lambda\rho} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}) + g_{\lambda\sigma} (g_{\mu\tau} g_{\nu\rho} - g_{\mu\rho} g_{\nu\tau}) + g_{\lambda\tau} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})], \\
\text{Tr}(\gamma^7 \sigma_{\lambda\mu\nu} \sigma_{\rho\sigma\tau}) &= 8 \epsilon_{\lambda\mu\nu\rho\sigma\tau}, \\
\text{Tr}(\gamma^7 \sigma_{\lambda\mu} \sigma_{\nu\rho} \sigma_{\sigma\tau}) &= -8i \epsilon_{\lambda\mu\nu\rho\sigma\tau}.
\end{aligned}$$

B.2 The 5-dimensional Galilean space-time

$$\begin{aligned}
\text{Tr}(\Gamma_{\mu_1} \cdots \Gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\Gamma_\mu \Gamma_\nu) &= 8\eta_{\mu\nu}, \\
\text{Tr}(\Gamma_\mu \Gamma_\nu \Gamma_\rho \Gamma_\sigma) &= 8(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}), \\
\text{Tr}(\Gamma^6) &= 0, \\
\text{Tr}(\Gamma^6 \Gamma_{\mu_1} \cdots \Gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\Gamma^6 \Gamma_\mu \Gamma_\nu) &= 0, \\
\text{Tr}(\Gamma^6 \Gamma_\mu \Gamma_\nu \Gamma_\rho \Gamma_\sigma) &= 0, \\
\text{Tr}(\Sigma_{\mu\nu}) &= 0, \\
\text{Tr}(\Sigma_{\mu\nu} \Sigma_{\rho\sigma}) &= 8(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}).
\end{aligned}$$

C Fierz identities

For later convenience, let us first introduce the following two quantities:

$$(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} = \frac{1}{4}[(\gamma^A)_{\alpha_1}{}^{\beta_1}(\gamma^B)_{\alpha_2}{}^{\beta_2} + (\gamma^A)_{\alpha_1}{}^{\beta_2}(\gamma^B)_{\alpha_2}{}^{\beta_1} + (\gamma^B)_{\alpha_1}{}^{\beta_2}(\gamma^A)_{\alpha_2}{}^{\beta_1} + (\gamma^B)_{\alpha_1}{}^{\beta_1}(\gamma^A)_{\alpha_2}{}^{\beta_2}], \quad (44)$$

$$[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} = \frac{1}{4}[(\gamma^A)_{\alpha_1}{}^{\beta_1}(\gamma^B)_{\alpha_2}{}^{\beta_2} - (\gamma^A)_{\alpha_1}{}^{\beta_2}(\gamma^B)_{\alpha_2}{}^{\beta_1} - (\gamma^B)_{\alpha_1}{}^{\beta_2}(\gamma^A)_{\alpha_2}{}^{\beta_1} + (\gamma^B)_{\alpha_1}{}^{\beta_1}(\gamma^A)_{\alpha_2}{}^{\beta_2}]. \quad (45)$$

From the definition in Eq. (44), we find the following properties:

$$\begin{aligned}
(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} &= (\gamma^B, \gamma^A)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2}, \\
(\gamma^A + \gamma^B, \gamma^C)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} &= (\gamma^A, \gamma^C)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} + (\gamma^B, \gamma^C)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2}, \\
(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}{}^{\gamma_1 \gamma_2}(\gamma^C, \gamma^D)_{\gamma_1 \gamma_2}{}^{\beta_1 \beta_2} &= \frac{1}{2}[(\gamma^A \gamma^C, \gamma^B \gamma^D)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} + (\gamma^A \gamma^D, \gamma^B \gamma^C)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2}].
\end{aligned}$$

Similarly, from Eq. (45) we find:

$$\begin{aligned}
[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} &= [\gamma^B, \gamma^A]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2}, \\
[\gamma^A + \gamma^B, \gamma^C]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} &= [\gamma^A, \gamma^C]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} + [\gamma^B, \gamma^C]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2}, \\
[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}{}^{\gamma_1 \gamma_2}[\gamma^C, \gamma^D]_{\gamma_1 \gamma_2}{}^{\beta_1 \beta_2} &= \frac{1}{2}([\gamma^A \gamma^C, \gamma^B \gamma^D]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} - [\gamma^A \gamma^D, \gamma^B \gamma^C]_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2}).
\end{aligned}$$

Also, the indices admit the following symmetry properties:

$$(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} = (\gamma^A, \gamma^B)_{\alpha_2 \alpha_1}{}^{\beta_1 \beta_2} = (\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}{}^{\beta_2 \beta_1}, \quad (46)$$

$$[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = -[\gamma^A, \gamma^B]_{\alpha_2 \alpha_1}^{\beta_1 \beta_2} = -[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\beta_2 \beta_1}. \quad (47)$$

The prime Fierz identity comes from the expansion of the identity operator $I_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}$ in terms of $(\gamma^A C)_{\alpha_1 \alpha_2}$ and $(C^{-1} \gamma^B)^{\beta_1 \beta_2}$:

$$I_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \sum_{A,B} C_A{}^B (\gamma^A C)_{\alpha_1 \alpha_2} (C^{-1} \gamma_B)^{\beta_1 \beta_2}. \quad (48)$$

The coefficients $C_A{}^B$ are determined by using the relation

$$(\gamma^A C)_{\alpha \beta} (C^{-1} \gamma_B)^{\beta \alpha} = \text{Tr}(\gamma^A \gamma_B) = 8 \delta_A{}^B. \quad (49)$$

With the operator $\frac{1}{64} (C^{-1} \gamma_A)^{\alpha_2 \alpha_1} (\gamma^B C)_{\beta_1 \beta_2}$ acting on Eq. (48), we obtain

$$C_A{}^B = \frac{1}{64} (C^{-1} \gamma_A)^{\alpha_2 \alpha_1} (\gamma^B C)_{\beta_1 \beta_2} \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \frac{1}{8} \delta_A{}^B.$$

Hence

$$\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \frac{1}{8} \sum_A (\gamma^A C)_{\alpha_1 \alpha_2} (C^{-1} \gamma_A)^{\beta_2 \beta_1}. \quad (50)$$

This is the prime Fierz identity in the $(5+1)$ Minkowski space-time.

Since the relationship (49) holds if we replace γ^A with Γ^A [see Eq. (25)], we can obtain the Fierz identity in the 5-dimensional Galilean space-time:

$$\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \frac{1}{8} \sum_A (\Gamma^A C)_{\alpha_1 \alpha_2} (C^{-1} \Gamma_A)^{\beta_2 \beta_1}. \quad (51)$$

It should be noted that the Galilean gamma matrices Γ^A are expressed in terms of γ^A s, so that the relationships (44) to (47) also hold for Γ^A s.

Recalling Eq. (8) and Eq. (9), we derive from the prime Fierz identity (50) that

$$\begin{aligned} \frac{1}{2} (\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} + \delta_{\alpha_1}^{\beta_2} \delta_{\alpha_2}^{\beta_1}) &= (\mathbb{I}, \mathbb{I})_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\ &= \frac{1}{8} [(C)_{\alpha_1 \alpha_2} (C^{-1})^{\beta_2 \beta_1} + \frac{1}{2} (\gamma^7 \sigma^{\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \gamma^7 \sigma_{\mu\nu})^{\beta_2 \beta_1} + \\ &\quad + \frac{1}{6} (\sigma^{\lambda\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \sigma_{\lambda\mu\nu})^{\beta_2 \beta_1}], \end{aligned} \quad (52)$$

$$\begin{aligned} \frac{1}{2} (\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_1}^{\beta_2} \delta_{\alpha_2}^{\beta_1}) &= [\mathbb{I}, \mathbb{I}]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\ &= \frac{1}{8} [(\gamma^7 C)_{\alpha_1 \alpha_2} (C^{-1} \gamma^7)^{\beta_2 \beta_1} + (\gamma^\mu C)_{\alpha_1 \alpha_2} (C^{-1} \gamma_\mu)^{\beta_2 \beta_1} + \\ &\quad + (i \gamma^7 \gamma^\mu C)_{\alpha_1 \alpha_2} (C^{-1} i \gamma^7 \gamma_\mu)^{\beta_2 \beta_1} + \frac{1}{2} (\sigma^{\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \sigma_{\mu\nu})^{\beta_2 \beta_1}]. \end{aligned} \quad (53)$$

Similarly, we obtain from Eq. (51), together with Eqs. (26) and (27), that

$$\begin{aligned} \frac{1}{2} (\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} + \delta_{\alpha_1}^{\beta_2} \delta_{\alpha_2}^{\beta_1}) &= (\mathbb{I}, \mathbb{I})_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\ &= \frac{1}{8} [(C)_{\alpha_1 \alpha_2} (C^{-1})^{\beta_2 \beta_1} + \frac{1}{2} (\Gamma^6 \Sigma^{\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \Gamma^6 \Sigma_{\mu\nu})^{\beta_2 \beta_1}], \end{aligned} \quad (54)$$

$$\begin{aligned}
\frac{1}{2}(\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_1}^{\beta_2}\delta_{\alpha_2}^{\beta_1}) &= [\mathbb{I}, \mathbb{I}]_{\alpha_1\alpha_2}^{\beta_1\beta_2}, \\
&= \frac{1}{8}[(\Gamma^6 C)_{\alpha_1\alpha_2}(C^{-1}\Gamma^6)^{\beta_2\beta_1} + (\Gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}\Gamma_\mu)^{\beta_2\beta_1} \\
&\quad + (i\Gamma^6\Gamma^\mu C)_{\alpha_1\alpha_2}(C^{-1}i\Gamma^6\Gamma_\mu)^{\beta_2\beta_1} + \frac{1}{2}(\Sigma^{\mu\nu}C)_{\alpha_1\alpha_2}(C^{-1}\Sigma_{\mu\nu})^{\beta_2\beta_1}].
\end{aligned} \tag{55}$$

Further Fierz identities follow from Eqs. (52) and (53):

$$\begin{aligned}
(\gamma^A, \gamma^B)_{\alpha_1\alpha_2}^{\beta_1\beta_2} &= \frac{1}{16}[\epsilon_B(\gamma^A\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\gamma^A C)_{\alpha_1\alpha_2}](C^{-1})^{\beta_2\beta_1} \\
&\quad + \frac{1}{32}[\epsilon_B(\gamma^A\gamma^\gamma\sigma^{\mu\nu}\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\gamma^\gamma\sigma^{\mu\nu}\gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\gamma^\gamma\sigma_{\mu\nu})^{\beta_2\beta_1} \\
&\quad + \frac{1}{96}[\epsilon_B(\gamma^A\sigma^{\lambda\mu\nu}\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\sigma^{\lambda\mu\nu}\gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\sigma_{\lambda\mu\nu})^{\beta_2\beta_1}, \\
[\gamma^A, \gamma^B]_{\alpha_1\alpha_2}^{\beta_1\beta_2} &= \frac{1}{16}[\epsilon_B(\gamma^A\gamma^\gamma\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\gamma^\gamma\gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\gamma^\gamma)^{\beta_2\beta_1} \\
&\quad + \frac{1}{16}[\epsilon_B(\gamma^A\gamma^\mu\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\gamma^\mu\gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\gamma_\mu)^{\beta_2\beta_1} \\
&\quad + \frac{1}{16}[\epsilon_B(\gamma^A i\gamma^\gamma\gamma^\mu\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B i\gamma^\gamma\gamma^\mu\gamma^A C)_{\alpha_1\alpha_2}](C^{-1}i\gamma^\gamma\gamma_\mu)^{\beta_2\beta_1} \\
&\quad + \frac{1}{32}[\epsilon_B(\gamma^A\sigma^{\mu\nu}\gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\gamma^B\sigma^{\mu\nu}\gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\sigma_{\mu\nu})^{\beta_2\beta_1}.
\end{aligned}$$

Similarly, from Eqs. (54) and (55), we have

$$\begin{aligned}
(\Gamma^A, \Gamma^B)_{\alpha_1\alpha_2}^{\beta_1\beta_2} &= \frac{1}{16}[\epsilon_B(\Gamma^A\Gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\Gamma^B\Gamma^A C)_{\alpha_1\alpha_2}](C^{-1})^{\beta_2\beta_1} \\
&\quad + \frac{1}{32}[\epsilon_B(\Gamma^A\Gamma^6\Sigma^{\mu\nu}\Gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\Gamma^B\Gamma^6\Sigma^{\mu\nu}\Gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\Gamma^6\Sigma_{\mu\nu})^{\beta_2\beta_1}, \\
[\Gamma^A, \Gamma^B]_{\alpha_1\alpha_2}^{\beta_1\beta_2} &= \frac{1}{16}[\epsilon_B(\Gamma^A\Gamma^6\Gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\Gamma^B\Gamma^6\Gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\Gamma^6)^{\beta_2\beta_1} \\
&\quad + \frac{1}{16}[\epsilon_B(\Gamma^A\Gamma^\mu\Gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\Gamma^B\Gamma^\mu\Gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\Gamma_\mu)^{\beta_2\beta_1} \\
&\quad + \frac{1}{16}[\epsilon_B(\Gamma^A i\Gamma^6\Gamma^\mu\Gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\Gamma^B i\Gamma^6\Gamma^\mu\Gamma^A C)_{\alpha_1\alpha_2}](C^{-1}i\Gamma^6\Gamma_\mu)^{\beta_2\beta_1} \\
&\quad + \frac{1}{32}[\epsilon_B(\Gamma^A\Sigma^{\mu\nu}\Gamma^B C)_{\alpha_1\alpha_2} + \epsilon_A(\Gamma^B\Sigma^{\mu\nu}\Gamma^A C)_{\alpha_1\alpha_2}](C^{-1}\Sigma_{\mu\nu})^{\beta_2\beta_1}.
\end{aligned}$$

D Explicit forms of the wave functions

In this appendix, we display the wave functions explicitly, in both the (5+1) Minkowski space-time (appendix D.1) and in the 5-dimensional Galilean space-time (appendix D.2).

D.1 The (5 + 1) Minkowski space-time

The wave functions are given by Eqs. (19) to (20). Their explicit expressions are given below.

$$u^{(1)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 1 \\ 0 \\ \frac{|\mathbf{p}|}{p^0 + m} n^4 \\ \frac{|\mathbf{p}|}{p^0 + m} (i + n^5) \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + i n^2 \end{pmatrix},$$

$$\bar{u}^{(1)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (1 + n^3, n^1 - i n^2) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 1, 0, -\frac{|\mathbf{p}|}{p^0 + m} n^4, -\frac{|\mathbf{p}|}{p^0 + m} (-i + n^5) \end{pmatrix},$$

$$u^{(2)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 1 \\ 0 \\ \frac{|\mathbf{p}|}{p^0 + m} n^4 \\ \frac{|\mathbf{p}|}{p^0 + m} (-i + n^5) \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} -n^1 + i n^2 \\ 1 + n^3 \end{pmatrix},$$

$$\bar{u}^{(2)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (-n^1 - i n^2, 1 + n^3) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 1, 0, -\frac{|\mathbf{p}|}{p^0 + m} n^4, -\frac{|\mathbf{p}|}{p^0 + m} (i + n^5) \end{pmatrix},$$

$$u^{(3)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 0 \\ 1 \\ \frac{|\mathbf{p}|}{p^0 + m} (-i + n^5) \\ -\frac{|\mathbf{p}|}{p^0 + m} n^4 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + i n^2 \end{pmatrix},$$

$$\bar{u}^{(3)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (1 + n^3, n^1 - i n^2) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 0, 1, -\frac{|\mathbf{p}|}{p^0 + m} (i + n^5), \frac{|\mathbf{p}|}{p^0 + m} n^4 \end{pmatrix},$$

$$u^{(4)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 0 \\ 1 \\ \frac{|\mathbf{p}|}{p^0 + m} (i + n^5) \\ -\frac{|\mathbf{p}|}{p^0 + m} n^4 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} -n^1 + i n^2 \\ 1 + n^3 \end{pmatrix},$$

$$\bar{u}^{(4)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (-n^1 - i n^2, 1 + n^3) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 0, 1, -\frac{|\mathbf{p}|}{p^0 + m} (-i + n^5), \frac{|\mathbf{p}|}{p^0 + m} n^4 \end{pmatrix},$$

$$v^{(1)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} \frac{|\mathbf{p}|}{p^0 + m} n^4 \\ \frac{|\mathbf{p}|}{p^0 + m} (-i + n^5) \\ 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} -n^1 + i n^2 \\ 1 + n^3 \end{pmatrix},$$

$$\bar{v}^{(1)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (-n^1 - i n^2, 1 + n^3) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} \frac{|\mathbf{p}|}{p^0 + m} n^4, \frac{|\mathbf{p}|}{p^0 + m} (i + n^5), -1, 0 \end{pmatrix},$$

$$v^{(2)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} -\frac{|\mathbf{p}|}{p^0 + m} n^4 \\ -\frac{|\mathbf{p}|}{p^0 + m} (\mathbf{i} + n^5) \\ -1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + \mathbf{i} n^2 \end{pmatrix},$$

$$\bar{v}^{(2)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (1 + n^3, n^1 - \mathbf{i} n^2) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} -\frac{|\mathbf{p}|}{p^0 + m} n^4, -\frac{|\mathbf{p}|}{p^0 + m} (-\mathbf{i} + n^5), 1, 0 \end{pmatrix},$$

$$v^{(3)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} \frac{|\mathbf{p}|}{p^0 + m} (\mathbf{i} + n^5) \\ -\frac{|\mathbf{p}|}{p^0 + m} n^4 \\ 0 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} -n^1 + \mathbf{i} n^2 \\ 1 + n^3 \end{pmatrix},$$

$$\bar{v}^{(3)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (-n^1 - \mathbf{i} n^2, 1 + n^3) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} \frac{|\mathbf{p}|}{p^0 + m} (-\mathbf{i} + n^5), -\frac{|\mathbf{p}|}{p^0 + m} n^4, 0, -1 \end{pmatrix},$$

$$v^{(4)}(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} -\frac{|\mathbf{p}|}{p^0 + m} (-\mathbf{i} + n^5) \\ \frac{|\mathbf{p}|}{p^0 + m} n^4 \\ 0 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + \mathbf{i} n^2 \end{pmatrix},$$

$$\bar{v}^{(4)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1 + n^3)}} (1 + n^3, n^1 - \mathbf{i} n^2) \otimes \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} -\frac{|\mathbf{p}|}{p^0 + m} (\mathbf{i} + n^5), \frac{|\mathbf{p}|}{p^0 + m} n^4, 0, 1 \end{pmatrix},$$

where we have used the notation

$$n^a = \frac{p^a}{|\mathbf{p}|}, \quad (a = 4, 5).$$

D.2 The 5-dimensional Galilean space-time

In this subsection, we give explicit forms of the wave functions given by Eqs. (42) to (43). The symbol P is a shorthand notation for the 5-momentum defined in Eq. (38).

$$u^{(1)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E + 3m}{E + m}} \begin{pmatrix} 1 \\ 0 \\ -\frac{E - m}{E + 3m} \\ \mathbf{i} 2 \sqrt{\frac{m(E - m)}{E + 3m}} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + \mathbf{i} n^2 \end{pmatrix},$$

$$\bar{u}^{(1)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (1+n^3, n^1-in^2) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(1, 0, \frac{E-m}{E+3m}, i2\frac{\sqrt{m(E-m)}}{E+3m} \right),$$

$$u^{(2)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 1 \\ 0 \\ -\frac{E-m}{E+3m} \\ -i2\frac{\sqrt{m(E-m)}}{E+3m} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1+in^2 \\ 1+n^3 \end{pmatrix},$$

$$\bar{u}^{(2)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (-n^1-in^2, 1+n^3) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(1, 0, \frac{E-m}{E+3m}, -i2\frac{\sqrt{m(E-m)}}{E+3m} \right),$$

$$u^{(3)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 0 \\ 1 \\ -i2\frac{\sqrt{m(E-m)}}{E+3m} \\ \frac{E-m}{E+3m} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1+in^2 \end{pmatrix},$$

$$\bar{u}^{(3)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (1+n^3, n^1-in^2) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(0, 1, -i2\frac{\sqrt{m(E-m)}}{E+3m}, -\frac{E-m}{E+3m} \right),$$

$$u^{(4)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 0 \\ 1 \\ i2\frac{\sqrt{m(E-m)}}{E+3m} \\ \frac{E-m}{E+3m} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1+in^2 \\ 1+n^3 \end{pmatrix},$$

$$\bar{u}^{(4)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (-n^1-in^2, 1+n^3) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(0, 1, i2\frac{\sqrt{m(E-m)}}{E+3m}, -\frac{E-m}{E+3m} \right),$$

$$v^{(1)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} -\frac{E-m}{E+3m} \\ -2i\frac{\sqrt{m(E-m)}}{E+3m} \\ 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1+in^2 \\ 1+n^3 \end{pmatrix},$$

$$\bar{v}^{(1)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (-n^1-in^2, 1+n^3) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(-\frac{E-m}{E+3m}, 2i\frac{\sqrt{m(E-m)}}{E+3m}, -1, 0 \right),$$

$$v^{(2)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} \frac{\frac{E-m}{E+3m}}{\frac{E-m}{E+3m}} \\ -2i \frac{\sqrt{m(E-m)}}{E+3m} \\ -1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1+in^2 \end{pmatrix},$$

$$\bar{v}^{(2)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (1+n^3, n^1-in^2) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(\frac{E-m}{E+3m}, 2i \frac{\sqrt{m(E-m)}}{E+3m}, 1, 0 \right),$$

$$v^{(3)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 2i \frac{\sqrt{m(E-m)}}{\frac{E+3m}{E-m}} \\ \frac{\frac{E+3m}{E-m}}{0} \\ 0 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1+in^2 \\ 1+n^3 \end{pmatrix},$$

$$\bar{v}^{(3)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (-n^1-in^2, 1+n^3) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(-2i \frac{\sqrt{m(E-m)}}{E+3m}, \frac{E-m}{E+3m}, 0, -1 \right),$$

$$v^{(4)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 2i \frac{\sqrt{m(E-m)}}{\frac{E+3m}{E-m}} \\ -\frac{\frac{E+3m}{E-m}}{0} \\ 0 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1+in^2 \end{pmatrix},$$

$$\bar{v}^{(4)}(P) = \frac{1}{\sqrt{2(1+n^3)}} (1+n^3, n^1-in^2) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left(-2i \frac{\sqrt{m(E-m)}}{E+3m}, -\frac{E-m}{E+3m}, 0, 1 \right).$$

It is important to remark that the solutions in subsection D have well-defined massless limits.

Acknowledgement

We acknowledge partial support by the Natural Sciences and Engineering Research Council of Canada. This manuscript was completed while M.K. was visiting the Theoretical Physics Institute at the University of Alberta. The authors wish to thank Profs. R. Jackiw and V.P. Nair for helpful comments and suggestions at the early stage of this work.

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